Target 25: Analyzing Game Theory in a Custom Dice Game

Independent Study and Mentorship Final Product

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Introduction

Game theory is the mathematical study of decision-making where the outcomes depend not only on one's own actions but also on the actions of others. It applies to a wide range of strategic situations, from economics and politics to biology and recreational games.

For this exploration, I designed a custom dice game called Target 25 to explore how game theory can be applied to model and predict player strategies under uncertainty. I further analyzed and calculated the probabilities associated with the winning conditions in a dice-rolling game. I explored different strategies that can improve the likelihood of success. In addition, I examined how changing variables such as the number of dice, the winning conditions, or the mechanics of the game can affect the probabilities.

Originally, I approached this exploration from a probability standpoint, aiming to calculate optimal strategies based on risk. However, as I observed player behavior, I realized that Target 25 provided a rich environment for studying interactive decision-making. Each player's decision not only influences their own chances of winning but also affects how an opponent navigates through different strategies. Thus, game theory became a natural and necessary lens through which to analyze the data.

The following terms will be used throughout this investigation.

- X represents the outcome of rolling **one** dice. Since a standard six-sided dice has possible outcomes of 1 through 6, $X \in \{1, 2, 3, 4, 5, 6\}$.
- Y represents the outcome of rolling **two** dice. Since the possible sums range from 2 to 12, $Y \in \{2, 3, 4, \ldots, 12\}$.

The probability of success is determined by the number of dice rolled, the sum of the dice rolls, the player's strategy, and the effect of penalties for exceeding 25 points.

For this exploration, I will design a custom dice game called "Target 25." The rules are as follows:

1. Each player starts at 0 points.
2. On their turn, a player may choose to roll either one or two six-sided dice.
3. The player can continue rolling until they decide to stop or exceed a total score of 25 points.
4. If a player exceeds 25 points, their score resets to zero.
5. The goal is to reach a score as close as possible to 25 without exceeding it. The highest valid score at the end of the game wins.
6. After one full game, the game is repeated where if a player exceeds 25 points, their score resets to zero, and they receive a penalty of 5 points deducted from their score in the following round.

The objective of this investigation is to evaluate how rational players make strategic decisions in a competitive setting with risk and uncertainty. By analyzing data collected from repeated rounds of gameplay, I aim to determine how different rule structures influence behavior, whether certain strategies emerge as dominant or optimal, and how closely real player decisions align with theoretical game theory models.

Probability Foundations and Risk Calculations

Before analyzing strategic interactions, it is important to understand the mathematical framework of the game.

For a single dice, the expected value is 3.5. Each number between 1 and 6 has a probability of:

$$P(X = k) = \frac{1}{6}$$

For two dice, the probability distribution becomes more complex as there are multiple combinations that can result in the same sum. Reference Figure 1 below.

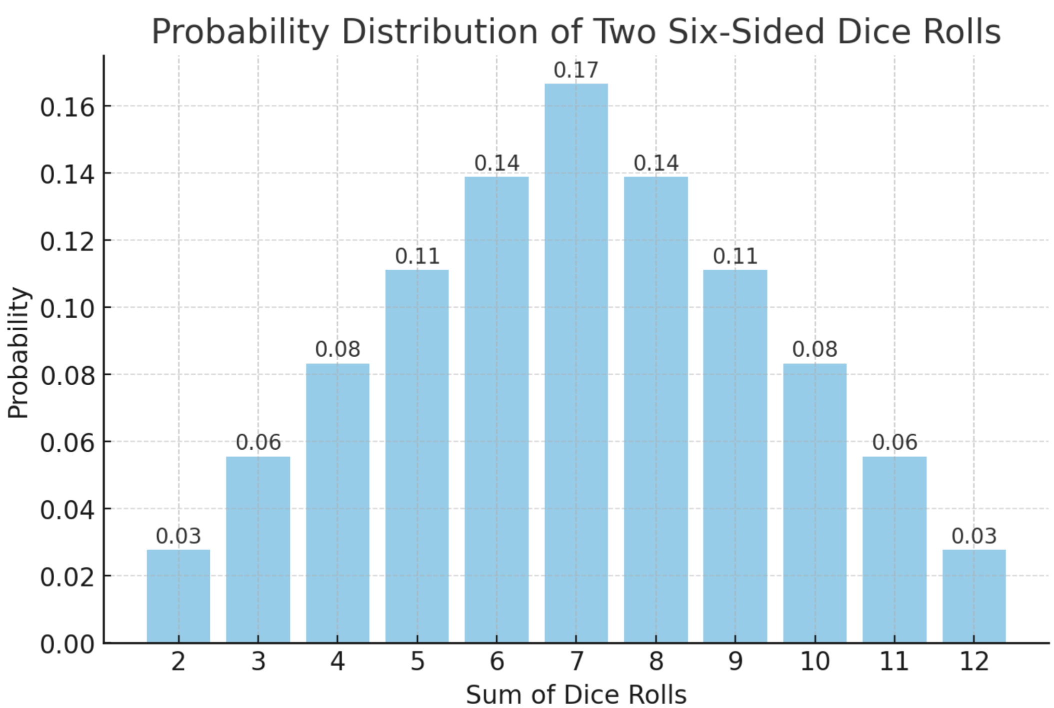

Figure 1: Probability Distribution of the Sum of Two Six-Sided Dice distribution

Considering that the order in which the dice land matters, I calculated the total number of outcomes as:

$$6 \text{ faces (first dice)} \times 6 \text{ faces (second dice)} = 36 \text{ possible outcomes}$$

Therefore, the probability to obtain a sum of 2, for example, is $\frac{1}{36}$, or 0.0278, proven in Figure 1.

To find the average outcome of an event over many trials, I found the expected value for rolling two dice:

$$E(Y) = \sum_{k=2}^{12}(x \times P(Y = k))$$

Using the probability distribution for two dice:

$$E(Y) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + \ldots + 12 \times \frac{1}{36}$$

$$E(Y) = \frac{252}{36} = 7$$

This means that on average, the sum of two dice rolls is 7.

In this dice game, the expected value estimates the average outcome of each roll; but, it does not give the probability of a specific outcome (such as reaching exactly 25 points or not exceeding 25). So, I decided to investigate conditional probability.

Let the current score be s. The probability of success $P(\text{success}|s)$ depends on the player's current score and the outcome of their next dice roll. For example, if the player has 22 points, the probability of exceeding 25 (and thereby failing) is higher than if they had, for example, 10 points. Thus, the player's decision-making is influenced by this probability.

If a player has 22 points, and they need to roll a dice, the probability of success depends on the number they roll. For example, if they roll a 3, they would exceed 25, but if they roll a 2 or less, they would succeed. This is visualized as:

$$P(\text{success}|s = 22) = P(\text{roll} \leq 3) = P(\text{roll} = 1) + P(\text{roll} = 2) + P(\text{roll} = 3)$$

For a single dice:

$$P(\text{success}|s=22) = P(X=1) + P(X=2) + P(X=3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

If the player rolls two dice, calculating the probabilities would change to find the probability of achieving a valid sum (less than or equal to 25) based on the two dice rolls.

For example, if the current score is s and a player rolls a sum k with two dice:

$$P(\text{success}|s,k) = P(\text{next score} \leq 25 \text{ and not exceeding } 25)$$

This involves considering the different sums and how likely it is that the player will reach a valid score (less than or equal to 25) without exceeding it.

Assuming the player has a score of 20. To calculate the probability of success for a roll of two dice, the steps are to:

- Identify all possible outcomes for a sum of 2 to 5 (rolling anything higher would exceed 25).
- Calculate the probability of the outcomes.
- Use conditional probability to determine the probability of success.

For example, the sum of two dice:

$$P(\text{success}|s=20) = P(\text{sum}=2) + P(\text{sum}=3) + P(\text{sum}=4) + P(\text{sum}=5)$$

From Figure 1, the probability of each sum is:

$$P(\text{sum}=2) = \frac{1}{36}, \quad P(\text{sum}=3) = \frac{2}{36}, \quad P(\text{sum}=4) = \frac{3}{36}, \quad P(\text{sum}=5) = \frac{4}{36}$$

So, the total probability of success is:

$$P(\text{success}|s=20) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} = \frac{10}{36} = \frac{5}{18}$$

Game Theory Concepts Applied to Target 25

The game results demonstrate several key concepts from game theory:

0.1 Dominant Strategies

A dominant strategy always results in the best outcome regardless of the opponent's actions. In Target 25, there is no strictly dominant strategy since optimal choices vary based on score, opponent's score, and remaining turns. The absence of a dominant strategy requires players to adopt mixed or contingent strategies.

0.2 Nash Equilibrium

A Nash equilibrium occurs when players choose strategies such that no one benefits from unilaterally changing their decision. In repeated rounds of Target 25, an equilibrium might emerge when both players adopt cautious strategies near a score of 20–22, reducing the likelihood of busting while maximizing the chance of winning. As observed in my data, this conservative approach became common once penalties were introduced.

0.3 Risk Dominance and Payoff Maximization

Players may choose strategies that maximize expected payoff but also consider minimizing the chance of extreme loss (like busting). This balance between risk and reward is central to game theory and was clearly observed as players adapted to penalties.

0.4 Minimax and Regret Minimization

Some players chose to stop early, even if a higher score was attainable, to avoid regret. This behavior reflects the minimax strategy, where players minimize the possible maximum loss.

It's often observed in decision-making under uncertainty, especially when emotional factors or penalties are involved.

Using the data from 10 full games played under two different rule conditions (with and without penalties), I analyzed how behavior changed and what that revealed about strategic adaptation.

0.5 Without Penalties (Appendix A)

Players frequently played aggressively, often continuing to roll even at high scores. Busting occurred frequently, as there was no long-term consequence. For instance, in Game 1, both players exceeded 25 in multiple rounds, resetting to zero and continuing without much adjustment in subsequent rounds.

0.6 With Penalties (Appendix B)

After introducing a -5 point penalty for busting, player behavior shifted significantly. Players who previously adopted aggressive strategies began to stop around 20–22. For example, in Game 2, Player B, who busted in the first round, changed their behavior in the next round, stopping early and ending with a winning score despite the penalty.

This change illustrates the influence of future consequences on current decision-making. The anticipation of future penalty reduced the short-term incentive to take high risks.

0.7 Strategic Adjustment Over Time

Over the 10 games, patterns emerged:

Players who experienced busts early began adjusting strategies to avoid future penalties.

Average final scores stabilized near 22–23, indicating an informal equilibrium.

Players began to mirror each other's strategies, possibly as a response to perceived optimal play.

These behaviors demonstrate the evolution of strategy over time and alignment with game theory predictions.

Conclusion

This investigation shows that even a simple dice game can produce complex strategic behavior modeled by game theory. The interaction between probability, risk, and adaptive behavior closely reflects principles seen in economics, finance, and behavioral science.

Key findings include:

Players do not always act based purely on probability; their behavior is shaped by past outcomes, opponent behavior, and future consequences.

Rule changes significantly affect strategy, revealing the importance of incentive structures.

Game theory provides a powerful lens for understanding how rational (and sometimes irrational) decisions evolve in competitive environments.

By integrating conditional probabilities with payoff analysis and strategic interaction, I gained a deeper understanding of both mathematical modeling and human behavior. The findings show clear real-world parallels, whether in investing, policymaking, or negotiations, strategic planning under uncertainty is a universal challenge.

Appendices

Appendix A

Game	Player	Rolls Taken	Final Score	Busts	Notes
1	A	7	28	Yes	Kept rolling past 25 despite the risk
1	B	5	26	Yes	Took aggressive risks and busted
2	A	5	24	No	Barely stopped in time
2	B	7	29	Yes	Ignored risks and busted
3	A	6	27	Yes	Overconfident and busted
3	B	6	25	No	Managed to hit exactly 25
4	A	7	30	Yes	Took unnecessary rolls and lost
4	B	6	26	Yes	Tried for 25 but miscalculated
5	A	5	25	No	Reached 25 successfully
5	B	7	28	Yes	Kept rolling past the limit
6	A	5	24	No	Stopped at a high but safe score
6	B	7	27	Yes	Took too many chances
7	A	6	26	Yes	Almost reached 25 but overdid it
7	B	7	30	Yes	Repeated mistakes from earlier games
8	A	5	25	No	Played aggressively but succeeded
8	B	7	28	Yes	Took a high-risk approach and lost
9	A	6	27	Yes	Rolled too many times
9	B	6	25	No	Managed to hit 25 exactly
10	A	7	31	Yes	Played recklessly to the end
10	B	6	26	Yes	Almost won but busted

Figure 2: Raw Data of Games without Penalties Organized in Google Sheets

Appendix B

Game	Player	Rolls Taken	Final Score	Busts	Penalty Applied	Notes
1	A	5	22	No	No	Stopped early to avoid busting
1	B	6	25	No	No	Took a risk and won the round
2	A	3	18	No	No	Played conservatively
2	B	4	23	No	No	Took extra rolls but stopped at a safe number
3	A	6	27	Yes	Yes (-5)	Overestimated chances and busted
3	B	5	19	No	No	Took measured rolls and stayed safe
4	A	5	20	No	No	Played cautiously after last bust
4	B	4	24	No	No	Stopped just before 25 to secure win
5	A	6	26	Yes	Yes (-5)	Took a risk but failed
5	B	5	21	No	No	Maintained a consistent strategy
6	A	4	22	No	No	Adjusted after previous bust
6	B	6	19	No	No	Aimed for a middle-ground score
7	A	5	25	No	No	Just barely won the round
7	B	6	27	Yes	Yes (-5)	Took too big of a risk
8	A	4	17	No	No	Played conservatively
8	B	5	22	No	No	Carefully managed rolls
9	A	6	26	Yes	Yes (-5)	Ignored past busts and repeated mistake
9	B	4	20	No	No	Played steadily without overcommitting
10	A	5	23	No	No	Took a slight risk but stayed under 25
10	B	6	24	No	No	Stopped just in time to avoid busting

Figure 3: Raw Data of Games with Penalties